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Paranormal Operators and Some Operator Equations

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Abstract. Let a pair (A, B) of bounded linear operators acting on a Hilbert space be a solution of the operator equations $ABA = A^2$ and $BAB = B^2$. When A is a paranormal operator, we explore some behaviors of the operators AB, BA, and B. In particular, if A or A^* is a polynomial root of paranormal operators, we show that Weyl type theorems are satisfied for the operators AB, BA, and B.

1. Introduction

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and let $B(\mathcal{H})$, $B_0(\mathcal{H})$ denote, respectively, the algebra of bounded linear operators, the ideal of compact operators acting on $\mathcal{H}$. If $T \in B(\mathcal{H})$, we shall write $N(T)$ and $R(T)$ for the null space and range of T. Also, let $\alpha(T) := \dim N(T)$, $\beta(T) := \dim N(T^*)$, and let $\sigma(T)$, $\sigma_a(T)$, $\sigma_s(T)$, $\sigma_p(T)$, $p_0(T)$, and $\pi_0(T)$ denote the spectrum, approximate point spectrum, surjective spectrum, point spectrum of T, the set of poles of the resolvent of T, and the set of all eigenvalues of T which are isolated in $\sigma(T)$, respectively. For $T \in B(\mathcal{H})$, the smallest nonnegative integer p such that $N(T^p) = N(T^{p+1})$ is called the *ascent* of T and denoted by $p(T)$. If no such integer exists, we set $p(T) = \infty$. The smallest nonnegative integer q such that $R(T^q) = R(T^{q+1})$ is called the *descent* of T and denoted by $q(T)$. If no such integer exists, we set $q(T) = \infty$.

Recall that $T \in B(\mathcal{H})$ is *hyponormal* if $T^*T \geq TT^*$ and T is *paranormal* if

$$\|Tx\|^2 \leq \|T^2x\|\|x\| \text{ for all } x \in \mathcal{H}.$$

An operator T is said to be *isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue of T, and T is called *normaloid* if $\|T\| = r(T)$, where $r(T)$ is the spectral radius of T.

It is well known that hyponormal operators imply paranormal operators, and paranormal operators entail a polynomial roots of paranormal operators. They are preserved under translation by scalars and under restriction to invariant subspaces. Moreover, it is easily shown that if $T \in B(\mathcal{H})$ is a polynomial root of paranormal operators, then it has SVEP from [1, Theorem 2.40]. The following facts follows from the above definition and some well known facts about paranormal operators.

(i) Every paranormal operator is isoloid and normaloid

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(ii) If T is paranormal and invertible, then T^{-1} is paranormal.
(iii) Every quasinilpotent paranormal operator is a zero operator.
(iv) T is paranormal if and only if $T^{2*}T^2 - 2\lambda T^*T + \lambda^2 \geq 0$ for all $\lambda > 0$.

Let (A, B) be a solution of the system of operator equations

$$ABA = A^2 \text{ and } BAB = B^2. \tag{1.1}$$

In [19], I. Vidav proved that A and B are self-adjoint operators satisfying the operator equations (1.1) if and only if $A = PP^*$ and $B = P^*P$ for some idempotent operator P. Also, the common spectral properties of the operators A and B satisfying the operator equations (1.1) have been studied by C. Schmoeger [17]. In particular, it is possible to relate the several spectrums, the single-valued extension property and Bishop's property (β) of A and B, which has been carried out by [12]. So, we are interested in the following question :

When A is paranormal, is it possible that the operator equations (1.1) preserve the properties of paranormal operators?

We start our program with the following section.

2. Preliminaries

An operator $T \in B(\mathcal{H})$ is called *upper semi-Fredholm* if it has closed range and finite dimensional null space and is called *lower semi-Fredholm* if it has closed range and its range has finite co-dimension. If $T \in B(\mathcal{H})$ is either upper or lower semi-Fredholm, then T is called *semi-Fredholm*, and *index of a semi-Fredholm operator* $T \in B(\mathcal{H})$ is defined by

$$i(T) := \alpha(T) - \beta(T).$$

If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called *Fredholm*. $T \in B(\mathcal{H})$ is called *Weyl* if it is Fredholm of index zero. For $T \in B(\mathcal{H})$ and a nonnegative integer n define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_0 = T$). If for some integer n the range $R(T^n)$ is closed and T_n is upper (resp. lower) semi-Fredholm, then T is called *upper* (resp. *lower*) *semi-B-Fredholm*. Moreover, if T_n is Fredholm, then T is called *B-Fredholm*. T is called *semi-B-Fredholm* if it is upper or lower semi-B-Fredholm.

Definition 2.1. Let $T \in B(\mathcal{H})$ and let $\Delta(T) := \{n \in \mathbb{N} : m \in \mathbb{N} \text{ and } m \geq n \Rightarrow (R(T^n) \cap N(T)) \subseteq (R(T^m) \cap N(T))\}$. Then the *degree of stable iteration* $\text{dis}(T)$ of T is defined as $\text{dis}(T) := \inf \Delta(T)$.

Let T be semi-B-Fredholm and let d be the degree of stable iteration of T. It follows from [8, Proposition 2.1] that T_m is semi-Fredholm and $i(T_m) = i(T_d)$ for each $m \geq d$. This enables us to define the *index of semi-B-Fredholm* T as the index of semi-Fredholm T_d. Let $BF(\mathcal{H})$ be the class of all B-Fredholm operators. In [5] they studied this class of operators and they proved [5, Theorem 2.7] that an operator $T \in B(\mathcal{H})$ is B-Fredholm if and only if $T = T_1 \oplus T_2$, where T_1 is Fredholm and T_2 is nilpotent. It appears that the concept of Drazin invertibility plays an important role for the class of B-Fredholm operators. Let $\mathcal{A}$ be a unital algebra. We say that an element $x \in \mathcal{A}$ is *Drazin invertible of degree k* if there exists an element $a \in \mathcal{A}$ such that

$$x^k a x = x^k, \ axa = a, \text{ and } xa = ax.$$

Let $a \in \mathcal{A}$. Then the *Drazin spectrum* is defined by

$$\sigma_D(a) := \{\lambda \in \mathbb{C} : a - \lambda \text{ is not Drazin invertible}\}.$$

It is well known that T is Drazin invertible if and only if it has finite ascent and descent, which is also equivalent to the fact that

$$T = T_1 \oplus T_2, \text{ where } T_1 \text{ is invertible and } T_2 \text{ is nilpotent.}$$

An operator $T \in B(\mathcal{H})$ is called *B-Weyl* if it is B-Fredholm of index 0. The *B-Fredholm spectrum* $\sigma_{BF}(T)$ and *B-Weyl spectrum* $\sigma_{BW}(T)$ of T are defined by

$$\sigma_{BF}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } B\text{-Fredholm}\},$$

$$\sigma_{BW}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } B\text{-Weyl}\}.$$

Now we consider the following sets:

$$\begin{aligned} BF_+(\mathcal{H}) &:= \{T \in B(\mathcal{H}) : T \text{ is upper semi-}B\text{-Fredholm}\},\\ BF_+^-(\mathcal{H}) &:= \{T \in B(\mathcal{H}) : T \in BF_+(\mathcal{H}) \text{ and } i(T) \le 0\},\\ LD(\mathcal{H}) &:= \{T \in B(\mathcal{H}) : p(T) < \infty \text{ and } R(T^{p(T)+1}) \text{ is closed}\}.\end{aligned}$$

By definition,

$$\sigma_{ea}(T) := \cap\{\sigma_a(T + K) : K \in B_0(\mathcal{X})\}$$

is the *essential approximate point spectrum*,

$$\sigma_{ab}(T) := \cap\{\sigma_a(T + K) : TK = KT \text{ and } K \in B_0(\mathcal{X})\}$$

is the *Browder essential approximate point spectrum*,

$$\sigma_{Bea}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin BF_+^-(\mathcal{H})\},$$

is the *upper semi-B-essential approximate point spectrum* and

$$\sigma_{LD}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin LD(\mathcal{H})\}$$

is the *left Drazin spectrum*. It is well known that

$$\sigma_{Bea}(T) \subseteq \sigma_{LD}(T) = \sigma_{Bea}(T) \cup \text{acc } \sigma_a(T) \subseteq \sigma_{BW}(T) \subseteq \sigma_D(T),$$

where we write acc K for the accumulation points of $K \subseteq \mathbb{C}$. If we write iso $K := K \setminus \text{acc } K$ then we let

$$\begin{aligned} \pi_{00}(T) &:= \{\lambda \in \text{iso } \sigma(T) : 0 < \alpha(T - \lambda) < \infty\},\\ \pi_{00}^a(T) &:= \{\lambda \in \text{iso } \sigma_a(T) : 0 < \alpha(T - \lambda) < \infty\},\\ p_{00}(T) &:= \sigma(T) \setminus \sigma_b(T),\\ p_{00}^a(T) &:= \sigma_a(T) \setminus \sigma_{ab}(T),\\ p_0^a(T) &:= \{\lambda \in \sigma_a(T) : T - \lambda \in LD(\mathcal{X})\}, \text{ and}\\ \pi_0^a(T) &:= \{\lambda \in \text{iso } \sigma_a(T) : \lambda \in \sigma_p(T)\}.\end{aligned}$$

We say that *Weyl's theorem holds for* $T \in B(\mathcal{H})$, in symbols ($\mathcal{W}$), if $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$, *Browder's theorem holds for* $T \in B(\mathcal{H})$, in symbols ($\mathcal{B}$), if $\sigma(T) \setminus \sigma_w(T) = p_{00}(T)$, *a-Weyl's theorem holds for* T , in symbols ($a\mathcal{W}$), if $\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}^a(T)$, and *a-Browder's theorem holds for* T, in symbols ($a\mathcal{B}$), if $\sigma_a(T) \setminus \sigma_{ea}(T) = p_{00}^a(T)$. The following variants of Weyl's theorem has been introduced in [7] and [8].

Definition 2.2. Let $T \in B(\mathcal{H})$.
(1) *Generalized Weyl's theorem holds for* T (in symbols, $T \in g\mathcal{W}$) if $\sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T)$.
(2) *Generalized Browder's theorem holds for* T (in symbols, $T \in g\mathcal{B}$) if $\sigma(T) \setminus \sigma_{BW}(T) = p_0(T)$.
(3) *Generalized a-Weyl's theorem holds for* T (in symbols, $T \in ga\mathcal{W}$) if $\sigma_a(T) \setminus \sigma_{Bea}(T) = \pi_0^a(T)$.
(4) *Generalized a-Browder's theorem holds for* T (in symbols, $T \in ga\mathcal{B}$) if $\sigma_a(T) \setminus \sigma_{Bea}(T) = p_0^a(T)$.

It is known ([7]) that the following relations hold:

$$\begin{array}{ccc} ga\text{-Weyl's theorem} & \Longrightarrow & ga\text{-Browder's theorem} \\ \Downarrow & & \Downarrow \\ g\text{-Weyl's theorem} & \Longrightarrow & g\text{-Browder's theorem} \\ \Downarrow & & \Downarrow \\ \text{Weyl's theorem} & \Longrightarrow & \text{Browder's theorem} \end{array}$$

In terms of local spectral theory ([1], [14]) recall that an important subspace $H_0(T)$ is the *quasi-nilpotent part* of T defined by

$$H_0(T) := \{x \in \mathcal{H} : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

If $T \in B(\mathcal{H})$, then the *analytic core* $K(T)$ is the set of all $x \in \mathcal{H}$ such that there exists a constant $c > 0$ and a sequence of elements $x_n \in \mathcal{H}$ such that $x_0 = x$, $Tx_n = x_{n-1}$, and $\|x_n\| \leq c^n\|x\|$ for all $n \in \mathbb{N}$. Given an arbitrary $T \in B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$, the *local resolvent set* $\rho_T(x)$ of T at the point $x \in \mathcal{H}$ is defined as the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $f : U \to \mathcal{H}$ which satisfies $(T - \lambda)f(\lambda) = x$ for all $\lambda \in U$. The *local spectrum* $\sigma_T(x)$ of T at the point $x \in \mathcal{H}$ is defined as $\sigma_T(x) := \mathbb{C} \setminus \rho_T(x)$. We define the *local spectral subspaces of* T by

$$H_T(F) := \{x \in \mathcal{H} : \sigma_T(x) \subseteq F\} \text{ for all sets } F \subseteq \mathbb{C}.$$

We say that $T \in B(\mathcal{H})$ has the *single valued extension property at* $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0) if for every open neighborhood U of λ_0 the only analytic function $f : U \longrightarrow \mathcal{H}$ which satisfies the equation

$$(T - \mu)f(\mu) = 0$$

is the constant function $f \equiv 0$ on U. The operator T is said to have SVEP if T has SVEP at every $\lambda_0 \in \mathbb{C}$. Evidently, every operator T, as well as its dual T^*, has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$, in particular, at every isolated point of $\sigma(T)$. We also have (see [1, Theorem 3.8])

$$p(T - \lambda) < \infty \implies T \text{ has SVEP at } \lambda, \tag{2.1}$$

and dually

$$q(T - \lambda) < \infty \implies T^* \text{ has SVEP at } \lambda. \tag{2.2}$$

It is well known from [1] that if $T-\lambda$ is semi-Fredholm, then the implications (2.1) and (2.2) are equivalent.

3. Main Results

Let a pair (A, B) *denote the solution of the operator equations* (1.1) *throughout this paper.* We explore some properties of a solution (A, B) of (1.1). In particular, when A is paranormal, B need not be a paranormal operator in general. For example, let $P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} I & 0 \\ I & 0 \end{pmatrix}$ in $B(\mathcal{H} \oplus \mathcal{H})$. Then $P^2 = P$ and $Q^2 = Q$. If $A := PQ$ and $B := QP$, then (A, B) is a solution of the operator equations (1.1). Since $B^* = \begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix}$, a straightforward calculation shows that

$$B^{2*}B^2 - 2\lambda B^*B + \lambda^2 I = \begin{pmatrix} (2 - 4\lambda + \lambda^2)I & 0 \\ 0 & \lambda^2 I \end{pmatrix},$$

But, $(2-4\lambda+\lambda^2)I$ is not a positive operator for $\lambda = 1$, hence we obtain that for some $\lambda > 0$,

$$B^{2*}B^2 - 2\lambda B^*B + \lambda^2 \ngeq 0.$$

Therefore B is neither paranormal nor normal. On the other hand, A is normal, so that it is a paranormal operator. From this, A is normaloid, however B need not be normaloid. In fact, $\sigma(B) = \{0, 1\}$, so that $r(B) = 1$. But, $\|B\| = \sqrt{2}$, hence B is not normaloid.

Let's consider another example. If $P = \begin{pmatrix} I & 2I \\ 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ are in $B(\mathcal{H} \oplus \mathcal{H})$, then both P and Q are idempotent operators. Also, $A := PQ$ and $B := QP$ satisfy the operator equations (1.1). Since $B^*A^* = \begin{pmatrix} I & 0 \\ 2I & 0 \end{pmatrix}$, a straightforward calculation shows that

$$(AB)^{2*}(AB)^2 - 2\lambda(AB)^*(AB) + \lambda^2 I = \begin{pmatrix} (1-2\lambda+\lambda^2)I & (2-4\lambda)I \\ (2-4\lambda)I & (4-8\lambda+\lambda^2)I \end{pmatrix}.$$

However, $(4-8\lambda+\lambda^2)I$ is not a positive operator for $\lambda = 1$, hence AB is neither paranormal nor normal. On the other hand, A is normal, so that it is a paranormal operator.

We now investigate some behaviors of the operators AB, BA and B whenever $A \in B(\mathcal{H})$ is a paranormal operator. We start with the following theorem.

Theorem 3.1. Let A be a paranormal operator on $\mathcal{H}$ and $N(A) = N(AB)$.
(1) If $\dim \mathcal{H} < \infty$, then AB is a normal operator.
(2) If $\dim \mathcal{H} < \infty$ and $N(A-\lambda) = N(B-\lambda)$ for each $\lambda \in \mathbb{C}$, then all of A, AB, BA, and B are normal operators.

Proof. Since $\sigma_p(AB) = \sigma_p(A)$ and $N(AB-\lambda) = N(A-\lambda)$ from [12],

$$\mathcal{K} := \sum_{\lambda\in\sigma_p(AB)} N(AB-\lambda) = \sum_{\lambda\in\sigma_p(A)} N(A-\lambda).$$

Since A is paranormal and $\dim \mathcal{H} < \infty$, it is known that $\mathcal{K}$ reduces A. So we can represent A as follows :

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} : \mathcal{K} \oplus \mathcal{K}^\perp \longrightarrow \mathcal{K} \oplus \mathcal{K}^\perp.$$

Assume that $\mathcal{K}^\perp \neq \{0\}$. Then $A_2 = A|\mathcal{K}^\perp$ is also paranormal. Since $\dim \mathcal{K}^\perp < \infty$, $\sigma_p(A_2) \neq \emptyset$. Thus for any $\lambda \in \sigma_p(A_2)$, there exists a nonzero $x_\lambda \in \mathcal{K}^\perp$ such that $\lambda x_\lambda = A_2 x_\lambda = A x_\lambda$. Hence $x_\lambda \in \mathcal{K}$. But, $x_\lambda \in \mathcal{K}^\perp$, hence $x_\lambda = 0$, which is a contradiction. Therefore $\mathcal{K}^\perp = \{0\}$, which implies that $\mathcal{H} = \mathcal{K}$. So for each $x \in \mathcal{H}$,

$$x = \sum_{\lambda\in\sigma_p(A)} x_\lambda = \sum_{\lambda\in\sigma_p(AB)} x_\lambda \quad \text{for some} \quad x_\lambda \in N(A-\lambda).$$

This implies that

$$ABx = \sum_{\lambda\in\sigma_p(AB)} \lambda x_\lambda = \sum_{\lambda\in\sigma_p(A)} \lambda x_\lambda = Ax.$$

On the other hand, since $A^*B^*A^* = A^{*2}$ and $B^*A^*B^* = B^{*2}$,

$$B^*A^*x = A^*x = \sum_{\lambda\in\sigma_p(A)} \bar{\lambda} x_\lambda = \sum_{\lambda\in\sigma_p(AB)} \bar{\lambda} x_\lambda.$$

Therefore

$$\|ABx\|^2 = \sum_{\lambda\in\sigma_p(AB)} \|\lambda x_\lambda\|^2 = \sum_{\lambda\in\sigma_p(AB)} |\lambda|^2 \|x_\lambda\|^2 = \sum_{\lambda\in\sigma_p(AB)} \|\bar{\lambda} x_\lambda\|^2 = \|B^*A^*x\|^2,$$

so that AB is normal. Thus (1) is valid. From [17] and $N(A-\lambda) = N(B-\lambda)$ for each $\lambda \in \mathbb{C}$, we note that

$$N(A-\lambda) = N(AB-\lambda) = N(BA-\lambda) = N(B-\lambda)$$

for every $\lambda \in \mathbb{C}$. Thus (2) is obvious by the similar process as above. □

Given $T \in B(\mathcal{H})$ and $S \in B(\mathcal{K})$ for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, the *commutator* $C(S,T) \in B(B(\mathcal{H},\mathcal{K}))$ is the mapping defined by

$$C(S,T)(A) := SA - AT \text{ for all } A \in B(\mathcal{H},\mathcal{K}).$$

The iterates $C(S,T)^n$ of the commutator are defined by $C(S,T)^0(A) := A$ and

$$C(S,T)^n(A) := C(S,T)(C(S,T)^{n-1}(A)) \text{ for all } A \in B(\mathcal{H},\mathcal{K}) \text{ and } n \in \mathbb{N};$$

they are often called the *higher order commutators*. There is the following binomial identity. It states that

$$C(S,T)^n(A) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k S^{n-k} A\, T^k,$$

which is valid for all $A \in B(\mathcal{H},\mathcal{K})$ and all $n \in \mathbb{N} \cup \{0\}$.

The following corollary illustrates that the higher order commutator equations $C(A,X)^n(A^*) = 0$ with all $n \in \mathbb{N}$ have a solution $\alpha AB + (1-\alpha)A$ for a real number α.

Corollary 3.2. Let A be paranormal with $N(A) = N(AB)$. If $\dim \mathcal{H} < \infty$ and α is a real number, then the following statements hold :
(1) $\alpha AB + (1-\alpha)A$ is a solution X of the operator equations $C(A,X)^n(A^*) = 0$ for all $n \in \mathbb{N}$.
(2) $\sigma_A(A^*x) \subseteq \sigma_{\alpha AB+(1-\alpha)A}(x)$ for all $x \in \mathcal{H}$.
(3) $A^*\mathcal{H}_{\alpha AB+(1-\alpha)A}(F) \subseteq \mathcal{H}_A(F)$ for every set F in $\mathbb{C}$.

Proof. (1) Since (A,B) is a solution of the operator equation $ABA = A^2$, it holds that $[\alpha AB + (1-\alpha)A]Y = YA$ where $Y := A$. By Theorem 3.1 it is known that AB and A are normal. Since $(\alpha AB)Y = \alpha ABA = \alpha A^2 = \alpha YA = Y(\alpha A)$ and AB is normal, it follows from Fuglede-Putnam theorem that $(\alpha AB)^*Y = Y(\alpha A)^*$, so that

$$[\alpha AB + (1-\alpha)A]^*Y = Y(\alpha A)^* + Y[(1-\alpha)A]^* = YA^*, \tag{3.1}$$

which implies that $C(A,X)^n(A^*) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k A^n A^* = 0$.

(2) If $\lambda_0 \notin \sigma_X(x)$, then there exists an analytic function $f : D \to \mathcal{H}$ defined on D a neighborhood of λ_0 such that $(X-\mu)f(\mu) \equiv x$ for every $\mu \in D$. So $A^*(X-\mu)f(\mu) \equiv A^*x$. It follows from (3.1) that $(A-\mu)A^*f(\mu) \equiv A^*x$. Therefore $\lambda_0 \in \rho_A(A^*x)$, so that $\lambda_0 \notin \sigma_A(A^*x)$.

(3) Let F be any set in $\mathbb{C}$ and $x \in \mathcal{H}_X(F)$ where $X = \alpha AB + (1-\alpha)A$ for real numbers α. Then $\sigma_X(x) \subseteq F$. From this part (2), $\sigma_A(A^*x) \subseteq F$. Therefore $A^*x \in \mathcal{H}_A(F)$. Hence we complete our proof. □

Proposition 3.3. The following statements are satisfied.
(1) Suppose $A \in B(\mathcal{H})$ is a paranormal weighted shift defined by $Ae_n = w_n e_{n+1}$ for $n = 0,1,2,\cdots$, where $w_n \neq 0$ for all $n \geq 1$. If $ABe_0 = w_0 e_1$, then AB is hyponormal.
(2) Suppose $B \in B(\mathcal{H})$ is a paranormal weighted shift defined by $Be_n = w_n e_{n+1}$ for $n = 0,1,2,\cdots$, where $w_n \neq 0$ for all $n \geq 1$. If $BAe_0 = w_0 e_1$, then BA is hyponormal.

Proof. Assume that A is a paranormal weighted shift defined by $Ae_n = w_n e_{n+1}$ for $n = 0,1,2,\cdots$. Then $\{|w_n|\}$ is an increasing sequence. Moreover,

$$w_n ABe_{n+1} = ABAe_n = A^2 e_n = w_n w_{n+1} e_{n+2},$$

so that $ABe_{n+1} = w_{n+1}e_{n+2}$ for $n = 0, 1, 2, \cdots$. But, $ABe_0 = w_0e_1$ and $|w_0| \leq |w_1|$, thus AB is hyponormal. So (1) is valid. Symmetrically, (2) is also satisfied. □

It is well known that every quasinilpotent paranormal operator is a zero operator. We apply this fact to a solution (A, B) of the operator equations (1.1).

Lemma 3.4. Let A be a paranormal operator and $\sigma(A) = \{\lambda\}$. Then the following statements hold.
(1) If $\lambda = 0$, then $B^2 = 0$.
(2) If $\lambda \neq 0$, then $\lambda = 1$ and $A = B = I$.

Proof. If $\lambda = 0$, then it follows from [10, Lemma 2.1] that $B^2 = 0$. So (1) is valid.

Suppose that $\lambda \neq 0$. Since A is paranormal, $A = \lambda I$. Since $ABA = A^2$, we have that $\lambda^2(B - I) = 0$, so that $B = I$. Also, if $BAB = B^2$, then $(\lambda - 1)B^2 = 0$ and $\lambda = 1$. Therefore $\sigma(A) = \sigma(B) = \{1\}$, which implies that $A = B = I$. □

From Lemma 3.4, we immediately have the following remark.

Remark 3.5. Let A be a paranormal operator. Then we have the following.
(1) If A is quasinilpotent, then AB, BA, and B are nilpotent.
(2) If $A - I$ is quasinilpotent, then B is the identity operator, that is, $AB - \lambda$, $BA - \lambda$, and $B - \lambda$ are invertible for all $\lambda \in \mathbb{C} \setminus \{1\}$.

Uchiyama [18] showed that if $T \in B(\mathcal{H})$ is a paranormal operator and λ_0 is an isolated point of $\sigma(T)$, then the Riesz idempotent $E_{\lambda_0}(T) := \frac{1}{2\pi i} \int_{\partial D} (\lambda - T)^{-1} d\lambda$, where D is the closed disk of center λ_0 which contains no other points of $\sigma(T)$, satisfies $R(E_{\lambda_0}(T)) = N(T - \lambda_0)$. Here, if $\lambda_0 \neq 0$, then $E_{\lambda_0}(T)$ is self-adjoint and $N(T - \lambda_0)$ reduces T. From this, we obtain the next corollary.

Corollary 3.6. If A is a paranormal operator, then iso $\sigma(T) \subseteq \{0, 1\}$ where $T \in \{A, AB, BA, B\}$.

Proof. Let λ_0 be a nonzero isolated point of $\sigma(A)$. Using the Riesz idempotent $E_{\lambda_0}(A)$ with respect to λ_0, we can represent A as the direct sum

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \text{ where } \sigma(A_1) = \{\lambda_0\} \text{ and } \sigma(A_2) = \sigma(A) \setminus \{\lambda_0\}.$$

Since A_1 is also paranormal, it follows from Lemma 3.4 that $\lambda_0 = 1$. This means that iso $\sigma(T) \subseteq \{0, 1\}$ where $T \in \{A, AB, BA, B\}$. □

Furthermore, we observe the following lemmas.

Lemma 3.7. If A is paranormal and λ_0 is a nonzero isolated point of $\sigma(AB)$, then for the Riesz idempotent $E_{\lambda_0}(A)$ with respect to λ_0, we have that

$$R(E_{\lambda_0}(A)) = N(AB - \lambda_0) = N(A^*B^* - \overline{\lambda_0}).$$

Proof. Since A is paranormal and $\lambda_0 \in \text{iso}\,\sigma(A) \setminus \{0\}$, by [18, Theorem 3.7], $R(E_{\lambda_0}(A)) = N(A - \lambda_0) = N(A^* - \overline{\lambda_0})$ for the Riesz idempotent $E_{\lambda_0}(A)$ with respect to λ_0. But, a pair (A, B) is a solution of the operator equations $ABA = A^2$ and $BAB = B^2$, hence by [17, Corollary 2.2],

$$N(A - \lambda_0) = N(AB - \lambda_0) \text{ and } N(A^* - \overline{\lambda_0}) = N(A^*B^* - \overline{\lambda_0}),$$

for $\lambda_0 \neq 0$. Therefore this completes the proof. □

Notation 3.8. We denote the set $\mathfrak{C}$ by the collection of every pair (A, B) of operators as the following:

$$\mathfrak{C} := \{(A, B) : A \text{ and } B \text{ are solutions of the operator equations (1.1) with } N(A-\lambda) = N(B-\lambda) \text{ for } \lambda \neq 0\}.$$

Then we have the following result.

Lemma 3.9. Suppose that $(A, B) \in \mathfrak{C}$ and A is paranormal. If $\lambda_0 \in \text{iso}\,\sigma(BA) \setminus \{0\}$, then for the Riesz idempotent $E_{\lambda_0}(A)$ with respect to λ_0, we have that

$$R(E_{\lambda_0}(A)) = N(BA - \lambda_0) = N(A^*B^* - \overline{\lambda_0}).$$

Proof. Since $(A, B) \in \mathfrak{C}$ and A is paranormal, it follows from [17, Corollary 2.2] and Lemma 3.7 that $N(BA - \lambda_0) = N(AB - \lambda_0) = N(A^*B^* - \overline{\lambda_0})$ for $\lambda_0 \in \text{iso}\,\sigma(BA) \setminus \{0\}$. Hence this completes the proof. □

From these arguments, we obtain the following result.

Proposition 3.10. Let $(A, B) \in \mathfrak{C}$ and A be a paranormal operator.
(1) If λ_0 is a nonzero isolated point of $\sigma(BA)$, then the range of $BA - \lambda_0$ is closed.
(2) If B^* is injective and $\lambda_0 \in \text{iso}\,\sigma(T) \setminus \{0\}$, then $N(T - \lambda_0)$ reduces T, where $T \in \{AB, B\}$.

Proof. (1) Let λ_0 be a nonzero isolated point of $\sigma(BA)$. Then it follows form Corollary 3.6 that iso $\sigma(BA) \subseteq \{1\}$. If iso $\sigma(BA) = \emptyset$, then it is obvious. Thus we only consider the case which 1 is an isolated point of $\sigma(BA)$. Since $ABA = A^2$ and $BAB = B^2$, by [17], 1 is an isolated point of $\sigma(A)$. Using the Riesz idempotent $E_1(A)$ with respect to 1, we can represent A as the direct sum

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \text{ where } \sigma(A_1) = \{1\} \text{ and } \sigma(A_2) = \sigma(A) \setminus \{1\}.$$

Since $(A, B) \in \mathfrak{C}$ and A is paranormal, by Lemma 3.9,

$$\mathcal{H} = R(E) \oplus R(E)^\perp = N(BA - I) \oplus N(BA - I)^\perp,$$

which implies that

$$BA = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}, \text{ where } \sigma(C_1) = \{1\} \text{ and } \sigma(C_2) = \sigma(BA) \setminus \{1\}.$$

Since A_1 and C_1 are the restrictions of A and BA to $R(E_1(A))$, respectively, we note that if $B_1 := B|R(E_1(A))$, then $A_1B_1A_1 = A_1^2$ and $B_1A_1B_1 = B_1^2$. Since A_1 is paranormal, it follows from Lemma 3.4 that $C_1 = I$. Thus

$$BA - I = 0 \oplus (C_2 - I),$$

so that

$$R(BA - I) = (BA - I)(\mathcal{H}) = 0 \oplus (C_2 - I)(N(BA - I)^\perp).$$

Since $C_2 - I$ is invertible, $BA - I$ has the closed range.

(2) Since a pair (A^*, B^*) is a solution of the operator equations $A^*B^*A^* = A^{*2}$ and $B^*A^*B^* = B^{*2}$ and B^* is injective, $A^*B^* = B^*$. But, $(A, B) \in \mathfrak{C}$, hence it follows from Lemma 3.7 and Lemma 3.9 that for the Riesz idempotent $E_{\lambda_0}(A)$,

$$R(E_{\lambda_0}(A)) = N(T - \lambda_0) = N(T^* - \overline{\lambda_0}),$$

where $T \in \{AB, B\}$. This completes the proof. □

It was shown by [13, Lemma 1] that for every $\lambda \in \pi_{00}(T)$, $\mathcal{H}_T(\{\lambda\})$ is finite dimensional if and only if $R(T - \lambda)$ is closed. Furthermore we can easily prove from [17] that

$$\pi_{00}(A) \setminus \{0\} = \pi_{00}(AB) \setminus \{0\} = \pi_{00}(BA) \setminus \{0\} = \pi_{00}(B) \setminus \{0\}.$$

Hence we have the following results from these arguments and Proposition 3.10.

Corollary 3.11. Let $(A, B) \in \mathfrak{C}$ and A be a paranormal operator. If $\lambda_0 \in \pi_{00}(BA) \setminus \{0\}$, then $\mathcal{H}_{BA}(\{\lambda_0\})$ is finite dimensional.

Remark 3.12. Let $(A, B) \in \mathfrak{C}$ and one of A, BA, AB, or B be paranormal. If λ_0 is a nonzero isolated point in the spectrum of one of them, then all of the ranges of $A - \lambda_0$, $BA - \lambda_0$, $AB - \lambda_0$, and $B - \lambda_0$ are closed. Moreover, if λ_0 is a nonzero isolated eigenvalue of the spectrum of one of them with finite multiplicity, then all of the spectral manifolds $\mathcal{H}_A(\{\lambda_0\})$, $\mathcal{H}_{AB}(\{\lambda_0\})$, $\mathcal{H}_{BA}(\{\lambda_0\})$, and $\mathcal{H}_B(\{\lambda_0\})$ are finite dimensional.

It is well known that every paranormal operators satisfy generalized Weyl's theorem [11], so that they have Weyl's theorem. Now, we would like to study that if A is paranormal, then Weyl's theorem holds for T, where $T \in \{AB, BA, B\}$. More generally, we study that if A or A^* is a polynomial root of paranormal operators, then generalized Weyl's theorem holds for $f(T)$ for $f \in H(\sigma(T))$, where $T \in \{AB, BA, B\}$. We start with the following lemma.

Lemma 3.13. We have the following properties :
(1) $\pi_0(A) = \pi_0(AB) = \pi_0(BA) = \pi_0(B)$.
(2) A is isoloid if and only if AB is isoloid if and only if BA is isoloid if and only if B is isoloid.

Proof. By [17] and [12, Lemma 2.3], it was known that $\sigma(A) = \sigma(AB) = \sigma(BA) = \sigma(B)$ and $\sigma_p(A) = \sigma_p(AB) = \sigma_p(BA) = \sigma_p(B)$. Thus (2) is valid. Also, it follows that for all $\lambda \in \mathbb{C}$,

$$\alpha(A - \lambda) > 0 \Leftrightarrow \alpha(AB - \lambda) > 0 \Leftrightarrow \alpha(BA - \lambda) > 0 \Leftrightarrow \alpha(B - \lambda) > 0,$$

which means that (1) is satisfied. □

Theorem 3.14. Suppose that A or A^* is a polynomial root of paranormal operators. Then $f(T) \in g\mathcal{W}$ for each $f \in H(\sigma(T))$, where $T \in \{AB, BA, B\}$.

Proof. Suppose that A is a polynomial root of paranormal operators. Let $T \in \{AB, BA, B\}$. We first show that T satisfies generalized Weyl's theorem. Suppose that $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Then $T - \lambda$ is B-Weyl but not invertible. It follows from [6, Lemma 4.1] that we can represent $T - \lambda$ as the direct sum

$$T - \lambda = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \text{ where } T_1 \text{ is Weyl and } T_2 \text{ is nilpotent.}$$

Since A is a polynomial root of paranormal operators, by [12, Theorem 2.1], T has SVEP. This implies that T_1 has SVEP at 0. However, T_1 is Weyl, hence T_1 has finite ascent and descent. From this, $T - \lambda$ has finite ascent and descent. So $\lambda \in \pi_0(T)$.

Conversely, suppose that $\lambda \in \pi_0(T)$. Then $\lambda \in \pi_0(A)$ by Lemma 3.13. But, A is a polynomial root of paranormal operators, hence $A \in g\mathcal{B}$ by [11, Theorem 4.14]. Therefore λ is a pole of the resolvent of A, so that $T - \lambda$ is Drazin invertible by [12, Theorem 2.11]. Thus we can represent $T - \lambda$ as the direct sum

$$T - \lambda = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \text{ where } T_1 \text{ is invertible and } T_2 \text{ is nilpotent.}$$

Therefore $T - \lambda$ is B-Weyl, and so $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Thus $\sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T)$, and hence $T \in g\mathcal{W}$.

Next we claim that $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$ for each $f \in H(\sigma(T))$. Since $T \in g\mathcal{W}$, $T \in g\mathcal{B}$. It follows from [11, Theorem 2.1] that $\sigma_{BW}(T) = \sigma_D(T)$. Since A is a polynomial root of paranormal operators, T has SVEP, so that $f(T)$ has SVEP for each $f \in H(\sigma(T))$. Hence $f(T) \in g\mathcal{B}$ by [11, Theorem 2.9]. Therefore we have that

$$\sigma_{BW}(f(T)) = \sigma_D(f(T)) = f(\sigma_D(T)) = f(\sigma_{BW}(T)).$$

Since A is a polynomial root of paranormal operators, it follows from [10, Lemma 2.3] that A is isoloid, equivalently, so is T by Lemma 3.13. From this, for each $f \in H(\sigma(T))$,

$$\sigma(f(T)) \setminus \pi_0(f(T)) = f(\sigma(T) \setminus \pi_0(T)).$$

Since $T \in g\mathcal{W}$, we have

$$\sigma(f(T)) \setminus \pi_0(f(T)) = f(\sigma(T) \setminus \pi_0(T)) = f(\sigma_{BW}(T)) = \sigma_{BW}(f(T)),$$

which implies that $f(T) \in g\mathcal{W}$.

Now we suppose that A^* is a polynomial root of paranormal operators. We first show that $T \in g\mathcal{W}$. Let $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Observe that $\sigma(T^*) = \overline{\sigma(T)}$ and $\sigma_{BW}(T^*) = \overline{\sigma_{BW}(T)}$. So $\overline{\lambda} \in \sigma(T^*) \setminus \sigma_{BW}(T^*)$. But, $A^*B^*A^* = A^{*2}$ and $B^*A^*B^* = B^{*2}$, hence $T^* \in g\mathcal{W}$. So $\overline{\lambda} \in p_0(T^*)$, which implies that $\overline{\lambda} \in p_0(A^*)$. Since A^* is a polynomial root of paranormal operators, $\overline{\lambda}$ is a pole of the resolvent of A^*, equivalently, λ is a pole of the resolvent of T. Thus $\lambda \in \pi_0(T)$.

Conversely, suppose $\lambda \in \pi_0(T)$. Then $\lambda \in \pi_0(A)$. Since $\lambda \in \text{iso}\ \sigma(A^*)$ and A^* is a polynomial root of paranormal operators, λ is a pole of the resolvent of A, so that $T - \lambda$ is Drazin invertible. Hence $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Thus $\sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T)$, so that $T \in g\mathcal{W}$. If A^* is a polynomial root of paranormal operators, then T is isoloid by Lemma 3.13. It follows from the first part of the proof that $f(T) \in g\mathcal{W}$. This completes the proof. □

Corollary 3.15. Suppose that $(A, B) \in \mathfrak{C}$ and A is a compact paranormal operator. Then we have that

$$BA = \begin{pmatrix} I & 0 \\ 0 & Q \end{pmatrix} \text{ on } N(BA - I) \oplus N(BA - I)^{\perp},$$

where Q is quasinilpotent.

Proof. Suppose that A is compact and paranormal. Then BA satisfies generalized Weyl's theorem by Theorem 3.14. Also, iso $\sigma(BA) \subseteq \{0, 1\}$ by Corollary 3.6. Thus it is satisfied that

$$\sigma(BA) \setminus \sigma_{BW}(BA) \subseteq \{0, 1\}. \tag{3.2}$$

Assume that $\sigma_{BW}(BA)$ is not finite. Then $\sigma(BA)$ is infinite from (3.2). Since A is compact, $\sigma(BA)$ is countable. Set $\sigma(BA) := \{0, \lambda_1, \lambda_2, \cdots\}$, where $\lambda_j \neq 0$ for $j = 1, 2, \cdots$, $\lambda_i \neq \lambda_j$ for every $i \neq j$, and $\lambda_j \to 0$ as $j \to \infty$. Then $\{\lambda_1, \lambda_2, \cdots\} \subseteq \text{iso}\ \sigma(BA) \setminus \{0\} \subseteq \{1\}$ by Corollary 3.6. But, this is a contradiction. Hence $\sigma_{BW}(BA)$ is finite. This means that every point in $\sigma_{BW}(BA)$ is isolated. So $\sigma(BA) \subseteq \{0, 1\}$. If $1 \notin \sigma(BA)$, then $\sigma(BA) = \{0\}$. Since A is paranormal, it follows from [10, Lemma 2,1] that $A = 0$, so that $BA = 0$. If $1 \in \sigma(BA)$, then by the proof of Proposition 3.10, $BA = I \oplus Q$ on $\mathcal{H} = N(BA - I) \oplus N(BA - I)^{\perp}$, where $\sigma(Q) = \{0\}$. This completes the proof. □

Now, we investigate that if A or A^* is a polynomial root of paranormal operators, then a-Browder's theorem holds for $f(T)$, where $f \in H(\sigma(T))$ and $T \in \{AB, BA, B\}$. For that, we first need the following lemma.

Lemma 3.16. Let $T \in \{AB, BA, B\}$. If A or A^* is a polynomial root of paranormal operators, then we have the following equalities for every $f \in H(\sigma(T))$.
(1) $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$ and
(2) $\sigma_w(f(T)) = f(\sigma_w(T))$.

Proof. Let $f \in H(\sigma(T))$. Since the inclusion $\sigma_{ea}(f(T)) \subseteq f(\sigma_{ea}(T))$ holds for every operator, it suffices to show the opposite inclusion. Suppose that $\lambda \notin \sigma_{ea}(f(T))$. Then $f(T) - \lambda$ is upper semi-Fredholm and $i(f(T) - \lambda) \leq 0$. Put

$$f(T) - \lambda = c(T - \mu_1)(T - \mu_2) \cdots (T - \mu_n) g(T),$$

where $c, \mu_1, \mu_2, \cdots, \mu_n \in \mathbb{C}$ and $g(T)$ is invertible. We note that if A is a polynomial root of paranormal operators, then it follows from [9, Corollary 2.10] and [1, Theorem 2.40] that it has SVEP. Hence T has SVEP by [12, Theorem 2.1]. Since $T - \mu_i$ is upper semi-Fredholm, it follows from [15, Proposition 2.1] that $i(T - \mu_i) \leq 0$ for each $i = 1, 2, \cdots, n$. So $\lambda \notin f(\sigma_{ea}(T))$.

Now, suppose that A^* is a polynomial root of paranormal operators. Since $A^*B^*A^* = A^{*2}$ and $B^*A^*B^* = B^{*2}$, T^* has also SVEP. So $i(T - \mu_i) \geq 0$ for each $i = 1, 2, \cdots, n$. From the classical index product theorem, $T - \mu_i$ is Weyl for each $i = 1, 2, \cdots, n$. Hence $\lambda \notin f(\sigma_{ea}(T))$, so that $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$. It follows that (1) is valid.

By the same argument as above, (2) is obtained. □

Theorem 3.17. Suppose that A or A^* is a root of paranormal operators. Then $f(T)$ satisfies a-Browder's theorem for every $f \in H(\sigma(T))$, where $T \in \{AB, BA, B\}$.

Proof. If A or A^* is a root of paranormal operators, then T or T^* has SVEP, so that a-Browder's theorem holds for T. Therefore by Lemma 3.16,

$$\sigma_{ab}(f(T)) = f(\sigma_{ab}(T)) = f(\sigma_{ea}(T)) = \sigma_{ea}(f(T)),$$

for every $f \in H(\sigma(T))$. □

Theorem 3.18. If A^* is a polynomial root of paranormal operators, generalized a-Weyl's theorem holds for T, where $T \in \{AB, BA, B\}$.

Proof. Suppose that A^* is a polynomial root of paranormal operators. Suppose that $\lambda \in \sigma_a(T) \setminus \sigma_{Bea}(T)$. Then $T - \lambda$ is upper semi-B-Fredholm and $i(T - \lambda) \leq 0$. Since $A^*B^*A^* = A^{*2}$ and $B^*A^*B^* = B^{*2}$, T^* has SVEP, so that $i(T - \lambda) \geq 0$. Thus $T - \lambda$ is B-Weyl, which implies that $\lambda \notin \sigma_{BW}(T)$. Since $T \in g\mathcal{B}$ by Theorem 3.14, $T - \lambda$ is Drazin invertible, so that λ is a pole of the resolvent of T. Hence $\lambda \in \text{iso}\,\sigma(T)$, which implies that $\lambda \in \text{iso}\,\sigma_a(T)$. Next we show that λ is an eigenvalue of T. Assume that $T - \lambda$ is injective. Since $R(T - \lambda)^{p(T-\lambda)+1}$ is closed and $p(T - \lambda) = 0$, we have that $T - \lambda$ has closed range. But, $T - \lambda$ is not bounded below, hence this is a contradiction. Therefore λ is an eigenvalue of T, so that $\lambda \in \pi_0^a(T)$.

Conversely, suppose that $\lambda \in \pi_0^a(T)$. Since T^* has SVEP, $\lambda \in \pi_0(T)$. Hence it follows from Theorem 3.14 that $T - \lambda$ is B-Weyl, so that $\lambda \in \sigma_a(T) \setminus \sigma_{BW}(T)$. But $\sigma_{Bea}(T) \subseteq \sigma_{BW}(T)$, hence $\lambda \in \sigma_a(T) \setminus \sigma_{Bea}(T)$. Thus $\pi_0^a(T) \subseteq \sigma_a(T) \setminus \sigma_{Bea}(T)$. Therefore $T \in ga\mathcal{W}$. □

Let $\mathcal{P}_0(\mathcal{H})$ denote the class of all operators $T \in B(\mathcal{H})$ such that there exists $p := p(\lambda) \in \mathbb{N}$ for which

$$H_0(T - \lambda) = N(T - \lambda)^p \text{ for all } \lambda \in \pi_{00}(T).$$

We construct $\mathcal{P}_1(\mathcal{H})$, contained in the set $\mathcal{P}_0(\mathcal{H})$, as the class of all operators $T \in B(\mathcal{H})$ such that there exists $p := p(\lambda) \in \mathbb{N}$ for which

$$H_0(T - \lambda) = N(T - \lambda)^p \text{ for all } \lambda \in \pi_0(T).$$

An operator $T \in B(\mathcal{H})$ is said to be *algebraic* if there exists a nontrivial polynomial h such that $h(T) = 0$. From the spectral mapping theorem it easily follows that the spectrum of an algebraic operator is a finite set. It is known that generalized Weyl's theorem is not generally transmitted to perturbation of operators satisfying generalized Weyl's theorem. In [2], they proved that if T is paranormal and F is an algebraic operator commuting with T, then Weyl's theorem holds for $T + F$. Throughout this motive we study that if A is a polynomial root of paranormal operators and F is an algebraic operator commuting with A and B, then generalized Weyl's theorem holds for $T + F$, where $T \in \{AB, BA, B\}$. We begin with the following lemma.

Lemma 3.19. We have the following equivalences :

$$A \in \mathcal{P}_1(\mathcal{H}) \Leftrightarrow AB \in \mathcal{P}_1(\mathcal{H}) \Leftrightarrow BA \in \mathcal{P}_1(\mathcal{H}) \Leftrightarrow B \in \mathcal{P}_1(\mathcal{H}).$$

Proof. Suppose that $A \in \mathcal{P}_1(\mathcal{H})$. We let $T \in \{AB, BA, B\}$ and $\lambda \in \pi_0(T)$. Since $ABA = A^2$ and $BAB = B^2$, by Lemma 3.13, $\lambda \in \pi_0(A)$. Then there exists $d \in \mathbb{N}$ such that $H_0(A-\lambda) = N(A-\lambda)^d$. Since $\lambda \in \text{iso}\,\sigma(A)$, by [1, Theorem 3.74], the analytic core $K(A-\lambda)$ is closed and

$$\mathcal{H} = K(A-\lambda) \oplus N(A-\lambda)^d.$$

Therefore we have that

$$(A-\lambda)^d(\mathcal{H}) = K(A-\lambda),$$

which implies by [1, Theorem 3.82] that λ is a pole of the resolvent of A with order d. Hence λ is also a pole of the resolvent of T with order d by [12, Theorem 2.11]. This means that $H_0(T-\lambda) = N(T-\lambda)^d$ for some $d \in \mathbb{N}$, so that $T \in \mathcal{P}_1(\mathcal{H})$. It is symmetrical that the converse holds. This completes the proof. □

Theorem 3.20. Let $T \in \{AB, BA, B\}$. Suppose that A is a polynomial root of paranormal operators and F is an algebraic operator commuting with A and B. Then $T + F \in g\mathcal{W}$.

Proof. Since A is a polynomial root of paranormal operators and F is algebraic, it is known that $T+F$ has SVEP from [3, Theorem 2.14]. To show that $T+F \in g\mathcal{W}$, we only need to prove that $T+F \in \mathcal{P}_1(\mathcal{H})$ by [4, Corollary 3.2]. Let $\lambda_0 \in \pi_0(T+F)$ and $\sigma(F) = \{\mu_1, \mu_2, \cdots, \mu_n\}$. The spectral decomposition provides a sequence of closed subspaces $\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_n$ which are invariant under F such that $\mathcal{H} = \oplus_{i=1}^n \mathcal{H}_i$ and $\sigma(F|\mathcal{H}_i) = \{\mu_i\}$ for each $i = 1, 2, \cdots, n$. Suppose that $E_{\mu_i}(F)$ are the corresponding spectral projections and $\mathcal{H}_i := R(E_{\mu_i}(F))$ for each $i = 1, 2, \cdots, n$. Since $\mathcal{H}_i = \{y \in \mathcal{H} : E_{\mu_i}(F)y = y\}$, we have that $E_{\mu_i}(F)(y_i) = y_i$ for arbitrary $y_i \in \mathcal{H}_i$. So if $S \in B(\mathcal{H})$ commutes with F, then $Sy_i = E_{\mu_i}(F)(Sy_i) \in \mathcal{H}_i$. Hence $\mathcal{H}_i$ is invariant under S for each $i = 1, 2, \cdots, n$. Now, let $T \in \{AB, BA, B\}$. Then $TF = FT$ and $\mathcal{H}_i$ is invariant under T as the argument above for each $i = 1, 2, \cdots, n$. Define $F_i := F|\mathcal{H}_i$, $B_i := B|\mathcal{H}_i$, and $A_i := A|\mathcal{H}_i$. Then $A_iB_iA_i = A_i^2$ and $B_iA_iB_i = B_i^2$. Since A_i is a polynomial root of paranormal operators, by [4, Theorem 2.8], $A_i \in \mathcal{P}_1(\mathcal{H}_i)$. It follows from Lemma 3.19 that $T_i \in \mathcal{P}_1(\mathcal{H}_i)$ for $T_i := T|\mathcal{H}_i$. So $T_i + \mu_i \in \mathcal{P}_1(\mathcal{H}_i)$. In fact, if $\gamma \in \pi_0(T_i + \mu_i)$, then $\gamma - \mu_i \in \pi_0(T_i)$. Since $T_i \in \mathcal{P}_1(\mathcal{H}_i)$, there exists a positive integer d such that $H_0(T_i + \mu_i - \gamma) = N(T_i + \mu_i - \gamma)^d$. Let h be a nonconstant complex polynomial such that $h(F) = 0$. Then $h(F_i) = h(F|H_i) = h(F)|H_i = 0$. From $\{0\} = \sigma(h(F_i)) = h(\sigma(F_i)) = h(\{\mu_i\})$, we have that $h(\mu_i) = 0$. Write $0 = h(F_i) = (F_i - \mu_i)^m g(F_i)$, where $g(F_i)$ is invertible. Hence $N_i := F_i - \mu_i$ are nilpotent for all $i = 1, 2, \cdots, n$. It follows from [4, Lemma 3.3] that

$$T_i + F_i = (T_i + \mu_i) + (F_i - \mu_i) = T_i + N_i + \mu_i \in \mathcal{P}_1(\mathcal{H}_i)$$

for every $i = 1, 2, \cdots, n$. Since $\lambda_0 \in \pi_0(T+F)$, if we fix $i \in \mathbb{N}$ such that $1 \le i \le n$, then $T_i + N_i - \lambda_0 + \mu_i = T_i + F_i - \lambda_0$ holds, so that we consider two cases :

Case I : Suppose that $T_i - \lambda_0 + \mu_i$ is invertible. Since N_i is a quasi-nilpotent operator commuting with $T_i - \lambda_0 + \mu_i$, it is clear that $T_i + F_i - \lambda_0$ is also invertible. Hence $H_0(T_i + F_i - \lambda_0) = N(T_i + F_i - \lambda_0) = \{0\}$.

Case II : Suppose that $T_i - \lambda_0 + \mu_i$ is not invertible. Then $\lambda_0 - \mu_i \in \sigma(T_i)$. We claim that $\lambda_0 \in \pi_0(T_i + F_i)$. Note that $\lambda_0 \in \sigma(T_i + \mu_i) = \sigma(T_i + F_i)$. Since $\sigma(T_i + F_i) \subseteq \sigma(T+F)$ and $\lambda_0 \in \text{iso}\,\sigma(T+F)$, $\lambda_0 \in \text{iso}\,\sigma(T_i + F_i)$. So we only prove that λ_0 is an eigenvalue of $T_i + F_i$. For that, we first show that $T_i + F_i - \lambda_0$ is B-Weyl. Since $N_i = F_i - \mu_i$, $\lambda_0 \in \text{iso}\,\sigma(T_i + N_i + \mu_i)$. Therefore $\lambda_0 - \mu_i \in \text{iso}\,\sigma(T_i + N_i) = \text{iso}\,\sigma(T_i)$, so that it follows from $A_iB_iA_i = A_i^2$ and $B_iA_iB_i = B_i^2$ that $\lambda_0 - \mu_i$ is an isolated point of $\sigma(A_i)$. Since A_i is a polynomial root of paranormal operators, $\lambda_0 - \mu_i \in p_0(A_i)$. This implies by $\sigma(A_i) = \sigma(T_i)$ and $\sigma_D(A_i) = \sigma_D(T_i)$ that $\lambda_0 - \mu_i \in \pi_0(T_i)$. By Theorem 3.14, generalized Weyl's theorem holds for T_i, which implies that $\lambda_0 - \mu_i \in \sigma(T_i) \setminus \sigma_{BW}(T_i)$. But N_i is nilpotent with $T_iN_i = N_iT_i$, hence $\sigma_D(T_i) = \sigma_D(T_i + N_i)$ and $T_i + N_i \in g\mathcal{B}$. Therefore we have $\sigma_{BW}(T_i + N_i) = \sigma_D(T_i + N_i)$. Hence

$$\pi_0(T_i) = \sigma(T_i) \setminus \sigma_{BW}(T_i) = \sigma(T_i + N_i) \setminus \sigma_{BW}(T_i + N_i).$$

Hence $T_i + F_i - \lambda_0$ is B-Weyl. Assume that $T_i + F_i - \lambda_0$ is injective. Then $\beta(T_i + F_i - \lambda_0) = \alpha(T_i + F_i - \lambda_0) = 0$, so that $T_i + F_i - \lambda_0$ is invertible. But, this is a contradiction from $\lambda_0 \in \sigma(T_i + F_i)$. Hence λ_0 is an eigenvalue of $T_i + F_i$, so that $\lambda_0 \in \pi_0(T_i + F_i)$. Since $T_i + F_i \in \mathcal{P}_1(\mathcal{H}_i)$, there exists a positive integer m_i such that $H_0(T_i + F_i - \lambda_0) = N(T_i + F_i - \lambda_0)^{m_i}$.

From Cases I and II we have

$$\begin{aligned} H_0(T+F-\lambda_0) &= \bigoplus_{i=1}^{n} H_0(T_i+F_i-\lambda_0) \\ &= \bigoplus_{i=1}^{n} N(T_i+F_i-\lambda_0)^{m_i} \\ &= N(T+F-\lambda_0)^m, \end{aligned}$$

where $m := \max\{m_1, m_2, \cdots, m_n\}$. Since λ_0 is arbitrary in $\pi_0(T+F)$, it follows that $T+F \in \mathcal{P}_1(\mathcal{H})$. Therefore this completes the proof. □

References

[1] P. Aiena, Fredholm and Local Spectral Theory, with Applications to Multipliers, Kluwer Academic Publishers, 2004.
[2] P. Aiena, *Weyl's theorem for perturbations of paranormal operators,* Proc. Amer. Math. Soc. **135** (2007), 2443–2451.
[3] P. Aiena, J.R. Guillen and P. Peña, *Property (w) for perturbations of polaroid operators,* Linear Algebra Appl. **428** (2008), 1791–1802.
[4] I.J. An and Y.M. Han, *Generalized Weyl's theorem for algebraically quasi-paranormal operators,* J. Ineq. Appl. **2012 : 89** (2012), 1–11.
[5] M. Berkani, *On a class of quasi-Fredholm operators,* Integr. Equ. Oper. Theory **34** (1999), 244–249.
[6] M. Berkani, *Index of B-Fredholm operators and generalization of a Weyl theorem,* Proc. Amer. Math. Soc. **130** (2002), 1717–1723.
[7] M. Berkani and J.J. Koliha, *Weyl type theorems for bounded linear operators,* Acta Sci. Math.(Szeged) **69** (2003), 359–376.
[8] M. Berkani and M. Sarih, *On semi B-Fredholm operators,* Glasgow Math. J. **43** (2001), 457–465.
[9] N.N. Chourasia and P.B. Ramanujan, *Paranormal operators on Banach spaces,* Bull. Austral. Math. Soc. **21** (1980), 161–168.
[10] R.E. Curto and Y.M. Han, *Weyl's theorem for algebraically paranormal operators,* Integr. Equ. Oper. Theory **47** (2003), 307–314.
[11] R.E. Curto and Y.M. Han, *Generalized Browder's and Weyl's theorems for Banach space operators,* J. Math. Anal. Appl. **336** (2007), 1424–1442.
[12] B.P. Duggal, *Operator equations $ABA = A^2$ and $BAB = B^2$,* Funct. Anal. Approx. Comput. **3** (2011), 9–18.
[13] S. Djordjević, I.H. Jeon and E. Ko, *Weyl's theoem throughout local spectral theory,* Glasgow Math. J. **44** (2002), 323–327.
[14] K.B. Laursen and M.M. Neumann, An Introduction to Local Spectral Theory, London Mathematical Society Monographs New Series 20, Clarendon Press, Oxford, 2000.
[15] M. Oudghiri, *Weyl's and Browder's theorem for operators satisfying the SVEP,* Studia Math. **163** (2004), 85–101.
[16] C. Schmoeger, *On the operator equations $ABA = A^2$ and $BAB = B^2$,* Publications de l'Institut Math.(N.S.) **78**(**92**) (2005), 127–133.
[17] C. Schmoeger, *Common spectral properties of linear operators A and B such that $ABA = A^2$ and $BAB = B^2$,* Publications de l'Institut Math.(N.S.) **79**(**93**) (2006), 109–114.
[18] A. Uchiyama, *On the isolated points of the spectrum of paranormal operators,* Integr. Equ. Oper. Theory **55** (2006), 145–151.
[19] I. Vidav, *On idempotent operators in a Hilbert space,* Publ. Inst. Math.(Beograd) **4**(**18**) (1964), 157–163.